Proof. Since $\varphi_\lambda(z)\in \mathrm{Exp}_{R(\lambda)}(\mathbf{C}_z^n)$, there exists a number $r < R(\lambda)$ such that $|u(z)|\leqslant M\exp r|z|$. Then, as follows from Cauchy's formula, for all $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $z\in\mathbf{C}^n$

$$|D^\alpha\varphi_\lambda(z)|\leqslant M_\lambda|\alpha|^{n/2}r^{|\alpha|}\exp r|z|, \tag{4.2}$$

where $M_\lambda > 0$ is a constant.

From this it follows that

$$A(D)u(z)\equiv\sum_\lambda\left(\sum_{|\alpha|=0}^{\infty}a_\alpha(\lambda)(D-\lambda I)^\alpha[e^{\lambda z}\varphi_\lambda(z)]\right)=\sum_\lambda e^{\lambda z}\left(\sum_{|\alpha|=0}^{\infty}a_\alpha(\lambda)D^\alpha\varphi_\lambda(z)\right)\equiv\sum_\lambda e^{\lambda z}\psi_\lambda(z),$$

where $\psi_\lambda(z)\in\mathrm{Exp}_{R(\lambda)}(\mathbf{C}_z^n)$, since by inequality (4.2)

$$|\psi_\lambda(z)|\leqslant M_\lambda\left(\sum_{|\alpha|=0}^{\infty}|a_\alpha(\lambda)||\alpha|^{n/2}r^{|\alpha|}\right)\exp r|z|\equiv M_{1\lambda}\exp r|z|$$

[we recall that $r < R(\lambda)$]. This means that $A(D)u(z)\in\mathrm{Exp}_\Omega(\mathbf{C}_z^n)$.

The continuity of the mapping (*) can be established by analogous and only a little more complex arguments. The theorem is proved.

From the theorem proved it follows that the collection of p/d operators A(D) with symbols $A(\zeta)\in\mathcal{O}(\Omega)$ and domain $\mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ form an algebra (the ring operation is composition). We denote this algebra by $\mathfrak{A}(\Omega)$. Further, let $\mathcal{O}(\Omega)$ be the algebra of analytic functions in the domain Ω.

The following conclusion is a corollary of Theorem 4.1.

Conclusion. The algebraic isomorphism

$$\mathfrak{A}(\Omega)\leftrightarrow\mathcal{O}(\Omega),$$

defined by the correspondence $A(D)\leftrightarrow A(\zeta)$ holds. Moreover, if together with $A(\zeta)$ the function $A^{-1}(\zeta)$ is also analytic in Ω, then

$$\frac{I}{A(D)}\circ A(D)=A(D)\circ\frac{I}{A(D)}=I,$$

where I is the identity operator.

Examples. 1. Let $u(z) = \exp\lambda z$ and let $A(\zeta)$ be analytic in a neighborhood of the point $\zeta = \lambda$. Then $A(D)\exp\lambda z = A(\lambda)\exp\lambda z$.

2. Let $n = 1$, $A(\zeta)=1/\zeta$, $\Omega=\mathbf{C}^1\setminus L$, where L is some ray issuing from the origin. Then

$$\mathrm{Exp}_\Omega(\mathbf{C}_z^1)=\left\{u(z):u(z)=\sum_\lambda e^{\lambda z}\varphi_\lambda(z)\right\},$$

where $\varphi_\lambda(z)\in\mathrm{Exp}_{R(\lambda)}(\mathbf{C}_z^1)$, and $R(\lambda)$ is the distance from the point λ to the ray L. The p/d operator I/D is the inverse operator to the operator of differentiation. Thus, to each function $u(z)\in\mathrm{Exp}_\Omega(\mathbf{C}_z^1)$ there is assigned the function

$$\frac{I}{D}u(z)=\sum_\lambda e^{\lambda z}\sum_{m=0}^{\infty}\frac{(-1)^m}{\lambda^{m+1}}D^m\varphi_\lambda(z),$$

which is the unique primitive function of u(z) which also belongs to the space $\mathrm{Exp}_\Omega(\mathbf{C}_z^n)$. We call this primitive "natural" and write

$$\frac{I}{D}u(z)=\mathrm{nat}\int u(z)\,dz.$$

5. Correctness of the Definition of a P/D Operator

Let Ω be a Runge domain. We shall show that the action of a p/d operator A(D) does not depend on the representation of a function $u(z)\in\mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ in the form of a sum

$$u(z)=\sum_\lambda e^{\lambda z}\varphi_\lambda(z), \tag{5.1}$$

where $\varphi_\lambda(z)\in\mathrm{Exp}_{R(\lambda)}(\mathbf{C}_z^n)$ (see Definition 4.1).

Indeed, suppose that in addition to (5.1) the function u(z) can be represented in the form

$$u(z)=\sum_{\mu} e^{\mu z}\hat{\varphi}_{\mu}(z), \tag{5.2}$$

where $\mu\in\Omega$, just as $\lambda\in\Omega$, runs through a finite set of values.

To prove that A(D)u(z) is well defined we use the multidimensional generalization of the Borel inversion formula (see, for example, [35]). Namely, if

$$u(z)=\sum_{|\alpha|=0}^{\infty} u_{\alpha}z^{\alpha}, \quad z\in\mathbf{C}^n,$$

is an entire function of exponential type $r = (r_1,\ldots,r_n)$, then

$$u(z)=\frac{1}{(2\pi i)^n}\int_{\Gamma_\varepsilon} Bu(\zeta)\,e^{z\zeta}d\zeta, \quad z\in\mathbf{C}^n,$$

where Γ_ε is the hull of the polycylinder $U_\varepsilon=\{\zeta:|\zeta_j|<r_j+\varepsilon,\ \varepsilon>0,\ j=1,\ldots,n\}$, and

$$Bu(\zeta)=\sum_{|\alpha|=0}^{\infty}\frac{\alpha!u_{\alpha}}{\zeta^{\alpha+1}} \quad (\alpha+1=(\alpha_1+1,\ldots,\alpha_n+1)).$$

In correspondence with this formula and the representation (5.1)

$$u(z)=\frac{1}{(2\pi i)^n}\sum_{\lambda} e^{\lambda z}\int_{\Gamma_{\varepsilon,\lambda}} B\varphi_{\lambda}(\zeta)\,e^{z\zeta}d\zeta, \tag{5.3}$$

where $B\varphi_\lambda(\zeta)$ is the function associated in the Borel sense with the function $\varphi_\lambda(\zeta)$, and $\Gamma_{\varepsilon,\lambda}$ is the hull of the polycylinder $U_{\varepsilon,\lambda}=\{\zeta:|\zeta_j|<r(\lambda)+\varepsilon,\ \varepsilon>0,\ j=1,\ldots,n\}$. We here choose the number $\varepsilon > 0$ such that translation of the hull $\Gamma_{\varepsilon,\lambda}$ by the vector λ lies strictly inside Ω. Then in correspondence with Definition 4.1 we obviously obtain

$$A(D)u(z)=\frac{1}{(2\pi i)^n}\sum_{\lambda}\left(\sum_{|\alpha|=0}^{\infty} a_{\alpha}(\lambda)(D-\lambda I)^{\alpha}e^{\lambda z}\int_{\Gamma_{\varepsilon,\lambda}} B\varphi_{\lambda}(\zeta)\,e^{z\zeta}d\zeta\right)=\frac{1}{(2\pi i)^n}\sum_{\lambda} e^{\lambda z}\int_{\Gamma_{\varepsilon,\lambda}} A(\lambda+\zeta)\,B\varphi_{\lambda}(\zeta)\,e^{z\zeta}d\zeta. \tag{5.4}$$

Similarly, for the representation (5.2) we obtain the formulas

$$u(z)=\frac{1}{(2\pi i)^n}\sum_{\mu} e^{\mu z}\int_{\Gamma_{\varepsilon,\mu}} B\hat{\varphi}_{\mu}(\zeta)\,e^{z\zeta}d\zeta, \quad z\in\mathbf{C}^n, \tag{5.5}$$

and

$$A(D)u(z)=\frac{1}{(2\pi i)^n}\sum_{\mu} e^{\mu z}\int_{\Gamma_{\varepsilon,\mu}} A(\mu+\zeta)\,B\hat{\varphi}_{\mu}(\zeta)\,e^{z\zeta}d\zeta \tag{5.6}$$

(the notation is clear).

To prove that the values of A(D)u(z) defined by formulas (5.4) and (5.6) coincide we consider the analytic functionals

$$L(v)\overset{\text{def}}{=}\frac{1}{(2\pi i)^n}\sum_{\lambda}\int_{\Gamma_{\varepsilon,\lambda}} v(\lambda+\zeta)\,B\varphi_{\lambda}(\zeta)\,d\zeta$$

and

$$M(v)\overset{\text{def}}{=}\frac{1}{(2\pi i)^n}\sum_{\mu}\int_{\Gamma_{\varepsilon,\mu}} v(\mu+\zeta)\,B\hat{\varphi}_{\mu}(\zeta)\,d\zeta,$$

where $v(\zeta)\in\mathcal{O}(\Omega)$ is an arbitrary function.

Formulas (5.3), (5.5) mean that for any $z\in\mathbf{C}^n$ the equality $L(e^{z\zeta})=M(e^{z\zeta})$ holds, whence it follows that the functionals L(v) and M(v) coincide on the set of all linear combinations of exponentials. Since Ω is a Runge domain, it follows that L(v) = M(v) for any function $v(\zeta)\in\mathcal{O}(\Omega)$. In particular, setting $v(\zeta)=A(\zeta)e^{z\zeta}$, we obtain the equality

$$L(A(\zeta)e^{z\zeta})=M(A(\zeta)e^{z\zeta}),$$

which by formulas (5.4) and (5.6) means that the definition of A(D)u(z) does not depend on the form of representing u(z). This is what was required.

In conclusion we shall show that the Runge condition is essential.

Counterexample. Let $\Omega = \mathbf{C}^1 \setminus \{0\}$, $A(D) = I/D$. Obviously, the symbol $A(\zeta) = 1/\zeta$ is analytic in Ω. We shall show that the operator A(D) is not single-valued in the space $\mathrm{Exp}_\Omega(\mathbf{C}_z^1)$. Indeed, for any $\lambda \neq 0$ we have by Cauchy's formula

$$e^{\lambda z} = \frac{1}{2\pi i} \int\limits_{|\eta|=R} \frac{e^{\eta z}}{\eta - \lambda} d\eta, \quad R > |\lambda|,$$

and hence, breaking the contour of integration into N sufficiently small parts, we find that

$$e^{\lambda z} = \sum_{j=0}^{N} \int\limits_{\Gamma_j} \frac{e^{\eta z}}{\eta - \lambda} d\eta \equiv \sum_{j=0}^{N} e^{\lambda_j z} \varphi_j(z), \quad \lambda_j \in \Gamma_j, \tag{5.7}$$

where

$$\varphi_j(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_j} \frac{e^{(\eta - \lambda_j) z}}{\eta - \lambda} d\eta.$$

Obviously, Γ_j can be taken so small that $|\varphi_j(z)| \leqslant M \exp r|z|$, where $r < |\lambda|$, $M > 0$ are constants. Thus, formula (5.7) gives a representation of the fucntion $e^{\lambda z}$ in the space $\mathrm{Exp}_\Omega(\mathbf{C}_z^1)$ in the form (5.1).

In correspondence with the definition of A(D)u(z) we have, on the one hand, $[I/D] \exp \lambda z = \lambda^{-1} \exp \lambda z$. On the other hand, proceeding from (5.7), we find that

$$\frac{I}{D} e^{\lambda z} = \frac{1}{2\pi i} \int\limits_{|\eta|=R} \frac{e^{\eta z} d\eta}{\eta(\eta - \lambda)} = \lambda^{-1}(\exp \lambda z - 1).$$

Thus, in the space $\mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ if Ω is not a Runge domain the operator A(D), generally speaking, may be multivalued.

In conclusion we note that an analogous representation holds for z^n, $n = 0, 1, \ldots$, i.e., $z^n \in \mathrm{Exp}_\Omega(\mathbf{C}_z^1)$, $\Omega = \mathbf{C}^1 \setminus \{0\}$, and $[I/D] z^n = z^{n+1}/(n+1)$.

6. Exponential Functionals

Definition 6.1. A continuous linear functional on the space $\mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ is called an exponential functional.

The space of all exponential functional we denote by $\mathrm{Exp}'_\Omega(\mathbf{C}_z^n)$ and call the space of exponential functionals associated with the domain Ω.

Example 1. $\delta(z)$ is obviously an exponential functional for any domain Ω.

Example 2. Let $A(\zeta) \in \mathcal{O}(\Omega)$. Then $h(z) = A(-D)\delta(z)$ $(-D = (-\partial/\partial z_1, \ldots, -\partial/\partial z_n))$ is an exponential functional acting on $v(z) \in \mathrm{Exp}_\Omega(\mathbf{C}_z^n)$ according to the formula

$$\langle h(z), v(z) \rangle \equiv \langle A(-D)\delta(z), v(z) \rangle \overset{\text{def}}{=} \langle \delta(z), A(D)v(z) \rangle.$$

We shall show that Example 2 exhausts all exponential functionals.

THEOREM 6.1. Let $h(z) \in \mathrm{Exp}'_\Omega(\mathbf{C}_z^n)$. Then there exists a unique function $A(\zeta) \in \mathcal{O}(\Omega)$ such that $h(z) = A(-D)\delta(z)$.

Proof. We set $A(\zeta) \equiv \langle h(z), \exp \zeta z \rangle$, $\zeta \in \Omega$. Obviously, $A(\zeta) \in \mathcal{O}(\Omega)$, and it is clear that

$$\langle A(-D)\delta(z), \exp \zeta z \rangle \overset{\text{def}}{=} \langle \delta(z), A(D) \exp \zeta z \rangle = A(\zeta),$$

i.e., the action of h(z) coincides with the action of the functional $A(-D)\delta(z)$ on functions $\exp \zeta z$, $\zeta \in \Omega$.

From the density lemma (Sec. 3) we conclude that the functionals h(z) and $A(-D)\delta(z)$ are equal as exponential functionals. The uniqueness of this representation is obvious. The theorem is proved.

Example 3. We consider the exponential functional $\mathscr{E}(z) = \exp(aD^2)\delta(z)$, where $a \in \mathbf{C}^n$ is a parameter, and $aD^2 = a_1 D_1^2 + \ldots + a_n D_n^2$. It can be shown (cf. [11, p. 110]) that for any function